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WALSH SPECTRAL ANALYSIS OF MULTIPLE DYADIC STATIONARY PROCESSES AND ITS APPLICATIONS

Takeaki NAGAI

Department of System Engineering, Faculty of Engineering, Oita University, Tannoharu 700, Oita 870, Japan

Masanobu TANIGUCHI

Department of Mathematics, Faculty of Science, Hiroshima University, Higashi-Sendamachi, Hiroshima 730, Japan

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In this paper, we investigate some properties of multiple dyadic stationary processes from the viewpoint of their Walsh spectral analysis. It is shown that under certain conditions a dyadic autoregressive and moving average process of finite order is expressed as a dyadic autoregressive process of finite order and also as a dyadic moving average process of finite order. We can see that the principal component process of such a dyadic stationary process has a simple finite structure in the sense that a dyadic filter which generates the principal component process has only one-side finite lags.

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Walsh spectral analysis * dyadic stationary process * finite parametric spectral model * principal component analysis * canonical correlation analysis

1. Introduction

Multiple random processes with covariance matrices invariant under dyadic addition are called dyadic stationary. In this paper, we show some peculiar properties of finite parametric linear models of dyadic stationary processes different from those of ordinary stationary processes (cf. Nagai (1977, 1980) and Morettin (1981) for univariate case and Taniguchi (1978) for the generalized case).

Analogously to the results in Nagai (1980, 1983, 1984), we shall establish, in Section 3, conditions such that multiple dyadic autoregressive and moving average processes of finite order can be expressed as a dyadic autoregressive process of finite order.

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In Section 4, we consider the principal component analysis of a multiple dyadic process with a Walsh spectral density matrix of rational type. In this case, we find that each principal component process is obtained as a one-sided finite sum of the observed process. For ordinary stationary processes, it is well-known that the rationality of a spectral density matrix does not always imply the one-side finiteness of terms generating the principal component processes (cf. Priestley, Rao and Tong (1973)).

In Section 5, we make a brief remark on the canonical correlation analysis of multiple dyadic processes.

2. Dyadic stationary processes

We denote by T the set of all non-negative integers. Let x and y be two non-negative real numbers and have the following binary expansions:

$$x = \sum_{l=-\infty}^{\infty} x_l 2^l, \quad \text{with } x_l = 0 \text{ or } 1, \qquad y = \sum_{l=-\infty}^{\infty} y_l 2^l, \quad \text{with } y_l = 0 \text{ or } 1.$$

Then, the dyadic addition $\oplus$ is defined by

$$x \oplus y = \sum_{l=-\infty}^{\infty} |x_l - y_l| 2^l.$$

We denote by $\{W(n, \lambda), 0 \leq \lambda \leq 1\}$, $n = 0, 1, \ldots$, the system of Walsh functions. The following properties of Walsh functions are well-known:

(i) For each $n \in T$, and $\lambda \in [0, 1]$, the value of $W(n, \lambda)$ is only $+1$ or -1.

(ii) For any $m, n \in T$,

$$W(n, \lambda) W(m, \lambda) = W(n \oplus m, \lambda), \quad \text{a.e. } \lambda.$$

(iii) For each $n \in T$, and $\lambda \in [0, 1]$,

$$W(n, \lambda) W(n, \mu) = W(n, \lambda \oplus \mu), \quad \text{a.e. } \mu.$$

(See Harmuth (1972) and Morettin (1974, 1981).)

Let $\{Y(t), t \in T\}$ be a q-dim. dyadic stationary process with zero mean vector and covariance matrices:

$$\Gamma_Y(h) = E\{Y(t) Y(t \oplus h)'\}, \quad h \in T.$$

Then, the Walsh spectral representations of $\{Y(t), t \in T\}$ and $\Gamma_Y(h)$, $h \in T$, are given by

$$Y(t) = \int_0^1 W(t, \lambda)\, \mathrm{d}Z_Y(\lambda) \quad \text{and} \quad \Gamma_Y(h) = \int_0^1 W(h, \lambda)\, \mathrm{d}F_Y(\lambda), \tag{2.1}$$

where $\{Z_Y(\lambda), 0 \leq \lambda \leq 1\}$ is a q-dim. Walsh spectral process with orthogonal increments and $F_Y(\lambda)$, $0 \leq \lambda \leq 1$, a $q \times q$ Walsh spectral matrix such that

$$E\{\mathrm{d}Z_Y(\lambda)\,\mathrm{d}Z_Y(\lambda)'\} = \mathrm{d}F_Y(\lambda),$$

(cf. Nagai (1977) and also Yaglom (1961) for general case).

The followings are the examples of dyadic stationary processes given by Morettin (1981).

Example 2.1. Let $\{U(t), t \in T\}$ be a q-dim. white noise process with zero mean vector and covariance matrices;

$$\begin{aligned}\Gamma_U(h) &= E\{U(t)U(t \oplus h)'\}\\ &= G \quad \text{for } h = 0,\\ &= 0 \quad \text{for } h \neq 0,\end{aligned}$$

where $G = \{g_{ij}\}$ is a $q \times q$ non-negative semi-definite matrix. Clearly, the white noise process is dyadic stationary. The Walsh spectral representation of the process $\{U(t), t \in T\}$ is given by

$$U(t) = \int_0^1 W(t, \lambda)\,\mathrm{d}Z_U(\lambda) \tag{2.2}$$

and its covariance matrices can be expressed as

$$\Gamma_U(h) = \int_0^1 W(h, \lambda) f_U(\lambda)\,\mathrm{d}\lambda$$

where the Walsh spectral density matrix $f_U(\lambda)$ is constant matrix G and

$$E\{\mathrm{d}Z_U(\lambda)\,\mathrm{d}Z_U(\lambda)'\} = f_U(\lambda)\,\mathrm{d}\lambda = G\,\mathrm{d}\lambda.$$

Example 2.2. Let $X(t) = \{X(t), t \in T\}$ be a superposition of N periodic "oscillations",

$$X(t) = \sum_{k=1}^{N} \xi_k W(t, x_k), \quad 0 \leq x_k \leq 1.$$

with ξ_i, $i = 1, 2, \ldots, N$, being random variables such that $E(\xi_i) = 0$, $E(\xi_i^2) = b_i$, $E(\xi_i \xi_j) = 0$, $i \neq j$, $i, j = 1, 2, \ldots, N$. Then

$$E(X(t)) = 0 \quad \text{and} \quad \Gamma_X(h) = \mathrm{Cov}(X(t), X(t \oplus h)) = \sum_{k=1}^{N} b_k W(h, x_k).$$

In particular, $\Gamma_X(0) = E(X(t)^2) = \sum_{k=1}^{N} b_k$, which shows that the average power of the composite oscillation is equal to the sum of the average powers of the periodic components.

3. Finite parametric linear models of dyadic stationary processes

It is in finite parametric linear models that the greatest differences between dyadic stationary processes and ordinary stationary processes occur. As a typical class of finite parametric linear models of dyadic stationary processes, we introduce autoregressive and moving average (DARMA) models of finite order. Dyadic autoregressive (DAR) models and dyadic moving average (DMA) models are considered as special cases of DARMA models.

For preparations, we consider properties of matrix valued functions expressed by finite linear combinations of Walsh functions. Let $\Phi(\lambda)=\{\phi_{ij}(\lambda)\}$ be a $q\times q$ matrix valued function of the form:

$$\Phi(\lambda)=\sum_{j=0}^{p} A_j W(j,\lambda), \quad 0\leq\lambda\leq 1. \tag{3.1}$$

where $p=2^m-1$ and $\{A_j\}_{j=0,1,2,\ldots,p}$ are $q\times q$ coefficient matrices. We call the function $\Phi(\lambda)$ defined by (3.1) *regular* if $\det[\Phi(\lambda)]\neq 0$, a.e. Let us denote by Σ a $q(p+1)\times q(p+1)$-block matrix whose (i,j)th block is $A_{(i-1)\oplus(j-1)}$, that is,

$$\Sigma=\begin{pmatrix} A_{0\oplus 0} & A_{0\oplus 1} & \cdots & A_{0\oplus p}\\ A_{1\oplus 0} & A_{1\oplus 1} & \cdots & A_{1\oplus p}\\ \vdots & \vdots & & \vdots\\ A_{p\oplus 0} & A_{p\oplus 1} & \cdots & A_{p\oplus p}\end{pmatrix}. \tag{3.2}$$

Lemma 3.1. *We have the following relation*:

$$\det[\Sigma]=\prod_{j=0}^{p}\det[\Phi(\lambda_j)], \tag{3.3}$$

where $\lambda_j=j/(p+1)$, $j=0,1,2,\ldots,p$.

Proof. Concerning a relation between the values of $\Phi(\lambda)$ and the coefficient matrices $\{A_j\}_{j=0,1,2,\ldots,p}$, we have the following:

$$\begin{aligned}\Sigma[H_{p+1}\oplus I_q]&=\Sigma\begin{pmatrix} W(0,\lambda_0)I_q & W(0,\lambda_1)I_q & \cdots & W(0,\lambda_p)I_q\\ W(1,\lambda_0)I_q & W(1,\lambda_1)I_q & \cdots & W(1,\lambda_p)I_q\\ \vdots & \vdots & & \vdots\\ W(p,\lambda_0)I_q & W(p,\lambda_1)I_q & \cdots & W(p,\lambda_p)I_q\end{pmatrix},\\ &=\left\{\sum_{l=0}^{p} A_{(i-1)\oplus l}W(l,\lambda_{j-1})\right\}_{(i,j)}=\{\Phi(\lambda_{j-1})W(i-1,\lambda_{j-1})\}_{(i,j)}\\ &=[H_{p+1}\otimes I_q]\operatorname{diag}[\Phi(\lambda_0)\quad \Phi(\lambda_1)\quad\cdots\quad\Phi(\lambda_p)],\end{aligned} \tag{3.4}$$

where I_q is the $q\times q$ identity matrix, H_{p+1} the Hadamard matrix of order $(p+1)$, and $\operatorname{diag}[\Phi(\lambda_0)\Phi(\lambda_1)\cdots\Phi(\lambda_p)]$ a $q(p+1)\times q(p+1)$ block diagonal matrix whose j-th $q\times q$ diagonal block is $\Phi(\lambda_{j-1})$. By notation $\Theta=\{\theta_{ij}\}_{(i,j)}$ used in (3.4), we mean a matrix Θ whose (i,j)th block element is θ_{ij}. Since $\det[H_{p+1}\otimes I_q]=N^{q(p+1)/2}\neq 0$, the equation (3.3) is directly obtained from (3.4). □

Corollary 3.1. *The matrix valued function $\Phi(\lambda)$ defined by* (3.1) *is regular if and only if* $\det[\Sigma] \neq 0$.

Proof. This is clear if we note that the function $\Phi(\lambda)$ takes at most $(p+1)$ different values, because

$$\Phi(\lambda) = \Phi(\lambda_j), \quad \text{for } \lambda \in [\lambda_j, \lambda_{j+1}),\ j = 0, 1, 2, \ldots, p. \qquad \square$$

Lemma 3.2. *Let the function $\Phi(\lambda)$ defined by* (3.1) *be regular. Then, there exists a matrix valued function $\eta(\lambda)$, which can be expressed as a finite linear combination of the Walsh functions*:

$$\eta(\lambda) = \sum_{l=0}^{p} B_l W(l, \lambda), \quad 0 \leq \lambda \leq 1, \tag{3.5}$$

and satisfies

$$\Phi(\lambda)\eta(\lambda) = I_q, \quad \text{a.e.} \tag{3.6}$$

where $q \times q$ coefficient matrices $\{B_l\}_{l=0,1,2,\ldots,p}$ are uniquely determined by the following equation:

$$\Sigma \begin{pmatrix} B_0 \\ B_1 \\ \vdots \\ B_p \end{pmatrix} = \begin{pmatrix} I_q \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{3.7}$$

In the above equation (3.7), *I_q is the $q \times q$ identity matrix and* 0 *is the $q \times q$ zero matrix.*

Proof. In order for the product of two functions $\Phi(\lambda)$ and $\eta(\lambda)$, i.e.

$$\Phi(\lambda)\eta(\lambda) = \sum_{l=0}^{p} \sum_{m=0}^{p} A_l B_m W(l \oplus m, \lambda) = \sum_{h=0}^{p} \Big[\sum_{j=0}^{p} A_{j \oplus h} B_j\Big] W(h, \lambda),$$

to be the $q \times q$ identity matrix I_q, the coefficient matrices $\{B_j\}_{j=0,1,\ldots,p}$ must satisfy

$$\sum_{h=0}^{p} \Big[\sum_{j=0}^{p} A_{j \oplus h} B_j\Big] W(h, \lambda_l) = I_q,$$

for all $l = 0, 1, 2, \ldots, p$. Using matrix notation, this is equivalently expressed as

$$(H'_{p+1} \otimes I_q) \begin{pmatrix} \sum_{j=0}^{p} A_j B_j \\ \sum_{j=0}^{p} A_{j \oplus 1} B_j \\ \vdots \\ \sum_{j=0}^{p} A_{j \oplus p} B_j \end{pmatrix} = \begin{pmatrix} I_q \\ I_q \\ \vdots \\ I_q \end{pmatrix}. \tag{3.8}$$

Since $(H_{p+1} \otimes I_q)(H'_{p+1} \otimes I_q) = (p+1) I_{q(p+1)}$, premultipling both sides of (3.8) (by $(H_{p+1} \otimes I_q)$, we obtain (3.7). $\square$

Now, as a class of finite parametric linear models of multiple dyadic stationary processes we define dyadic autoregressive moving average (DARMA) processes of finite order. As special cases of them, we also define dyadic autoregressive (DAR) processes and dyadic moving average (DMA) processes either of finite orders.

Definition 3.1. We call a q-dim. dyadic stationary process $\{Y(t), t \in T\}$ a dyadic autoregressive moving average process of order (p, r), denoted by DARMA(p, r), if it can be expressed as

$$\sum_{n=0}^{p} A_n Y(n \oplus t) = \sum_{l=0}^{r} B_l U(l \oplus t), \quad t = 0, 1, 2, \ldots, \tag{3.9}$$

where (i) p and r are non-negative integers of the forms $p = 2^m - 1$, and $r = 2^f - 1$, respectively (ii) $\{U(t), t \in T\}$ is a q-dim. white noise process and (iii) coefficients $\{A_n\}_{n=0,1,2,\ldots,p}$ and $\{B_l\}_{l=0,1,2,\ldots,r}$ are $q \times q$ square matrices and there are at least two non-zero matrices A_{n_0} and B_{l_0} for some n_0, $2^{m-1} \leq n_0 \leq 2^m - 1$ and l_0, $2^{f-1} \leq l_0 \leq 2^f - 1$.

In particular, if $r = 0$, we call it a dyadic autoregressive process of order p, denoted by DAR(p), and if $p = 0$, a dyadic moving average process of order r, denoted by DMA(r). Let us put $k = \max(p, r)$. And we denote by Σ a $q(p+1) \times q(p+1)$ matrix defined by (3.2) and by S a $q(r+1) \times q(r+1)$ matrix with $B_{(i-1)\oplus(j-1)}$ as its (i, j)th $q \times q$ block; that is,

$$S = \begin{pmatrix} B_{0\oplus 0} & B_{0\oplus 1} & \cdots & B_{0\oplus r} \\ B_{1\oplus 0} & B_{1\oplus 1} & \cdots & B_{1\oplus r} \\ \vdots & \vdots & & \vdots \\ B_{r\oplus 0} & B_{r\oplus 1} & \cdots & B_{r\oplus r} \end{pmatrix}.$$

From Lemma 3.2, if $\det(\Sigma) \neq 0$, there exists a $q \times q$ matrix valued function

$$\eta_1(\lambda) = \sum_{n=0}^{p} D_n W(n, \lambda)$$

which is an inverse matrix of

$$\Phi_1(\lambda) = \sum_{n=0}^{p} A_n W(n, \lambda);$$

that is, $\eta_1(\lambda) = \Phi_1(\lambda)^{-1}$. Similarly, if $\det(S) \neq 0$, then there is a $q \times q$ matrix valued function

$$\eta_2(\lambda) = \sum_{n=0}^{r} L_n W(n, \lambda)$$

which is an inverse matrix of

$$\Phi_2(\lambda) = \sum_{n=0}^{r} B_n W(n, \lambda);$$

that is, $\eta_2(\lambda) = \Phi_2(\lambda)^{-1}$.

Now we have the following theorem:

Theorem 3.1. *Let* $\{Y(t), t \in T\}$ *be a* q-dim. *dyadic autoregressive moving average process of order* (p, r) *defined by* (3.9). *Then, we have the following results:*

(i) *If* $\det(\Sigma) \neq 0$, *then* $\{Y(t), t \in T\}$ *is a* DMA(k)*-process which may be written as*

$$Y(t) = \sum_{n=0}^{k} K_n U(n \oplus t), \quad t \in T, \tag{3.10}$$

where

$$\begin{aligned} K_n &= \sum_{s=0}^{p} D_s B_{s \oplus n}, \quad n = 0, 1, 2, \ldots, k, \quad \textit{if } p \leq r, \\ &= \sum_{s=0}^{r} D_{s \oplus n} B_s, \quad n = 0, 1, 2, \ldots, k, \quad \textit{if } r < p. \end{aligned} \tag{3.11}$$

(ii) *If* $\det(S) \neq 0$, *then* $\{Y(t), t \in T\}$ *is a* DAR(k)*-process satisfying the equation*

$$\sum_{s=0}^{k} C_s Y(t \oplus s) = U(t), \quad t \in T, \tag{3.12}$$

where

$$\begin{aligned} C_s &= \sum_{j=0}^{r} L_j A_{j \oplus s}, \quad s = 0, 1, 2, \ldots, k, \quad \textit{if } r \leq p, \\ &= \sum_{j=0}^{p} L_{j \oplus s} A_j, \quad s = 0, 1, 2, \ldots, k, \quad \textit{if } r > 0. \end{aligned} \tag{3.13}$$

Proof. Suppose that the Walsh spectral representations of $Y(t)$ and $U(t)$ are given by (2.1) and (2.2) respectively. Then, the equation (3.9) is written as follows:

$$\begin{aligned} \sum_{n=0}^{p} A_n Y(n \oplus t) &= \int_0^1 W(t, \lambda) \Phi_1(\lambda)\, \mathrm{d}Z_Y(\lambda) = \sum_{l=0}^{r} B_l U(t \oplus l) \\ &= \int_0^1 W(t, \lambda) \Phi_2(\lambda)\, \mathrm{d}Z_U(\lambda), \quad t \in T. \end{aligned} \tag{3.14}$$

Since the system of Walsh functions $\{W(t, \lambda), 0 \leq \lambda \leq 1\}$, $t \in T$, is complete and the relation (3.14) holds for all $t \in T$, we have from (3.14) that

$$\Phi_1(\lambda)\, \mathrm{d}Z_Y(\lambda) = \Phi_2(\lambda)\, \mathrm{d}Z_U(\lambda), \quad 0 \leq \lambda \leq 1. \tag{3.15}$$

(i) Suppose that $\det(\Sigma) \neq 0$. Then, by premultiplying both sides of (3.15) by $\Phi_1^{-1}(\lambda) = \eta_1(\lambda)$, we have

$$\begin{aligned} \mathrm{d}Z_Y(\lambda) &= \eta_1(\lambda)\Phi_2(\lambda)\,\mathrm{d}Z_U(\lambda) \\ &= \sum_{j=0}^{k}\left[\sum_{l=0}^{p} D_l B_{l\oplus j}\right] W(j,\lambda)\,\mathrm{d}Z_U(\lambda) \quad \text{if } p \le r \\ &= \sum_{j=0}^{k}\left[\sum_{l=0}^{r} D_{l\oplus j} B_l\right] W(j,\lambda)\,\mathrm{d}Z_U(\lambda) \quad \text{if } r < p \\ &= \sum_{j=0}^{k} K_j W(j,\lambda)\,\mathrm{d}Z_U(\lambda), \quad 0 \le \lambda \le 1, \end{aligned} \tag{3.16}$$

where coefficients $\{K_j\}_{j=0,1,2,\ldots,k}$ are given by (3.11). From (3.16), it follows that

$$Y(t) = \int_0^1 W(t,\lambda)\,\mathrm{d}Z_Y(\lambda) = \sum_{j=0}^{k} K_j U(t \oplus j), \quad t \in T,$$

and thus the equation (3.10) is established.

(ii) Similarly, if $\det(S) \neq 0$, then by premultiplying both sides of (3.15) by $\Phi_2^{-1}(\lambda) = \eta_2(\lambda)$, we have

$$\begin{aligned} \mathrm{d}Z_U(\lambda) &= \eta_2(\lambda)\Phi_1(\lambda)\,\mathrm{d}Z_Y(\lambda) \\ &= \sum_{l=0}^{k}\left[\sum_{j=0}^{p} L_{j\oplus l} A_j\right] W(l,\lambda)\,\mathrm{d}Z_Y(\lambda) \quad \text{if } p \le r \\ &= \sum_{l=0}^{k}\left[\sum_{j=0}^{r} L_j A_{j\oplus l}\right] W(l,\lambda)\,\mathrm{d}Z_Y(\lambda) \quad \text{if } r < p \\ &= \sum_{l=0}^{k} C_l W(l,\lambda)\,\mathrm{d}Z_Y(\lambda), \quad 0 \le \lambda \le 1, \end{aligned} \tag{3.17}$$

where coefficients $\{C_l\}_{l=0,1,2,\ldots,k}$ are given by (3.13). From (3.17), it follows that

$$U(t) = \int_0^1 W(t,\lambda)\,\mathrm{d}Z_U(\lambda) = \sum_{l=0}^{k} C_l Y(t \oplus l), \quad t \in T,$$

and thus it is shown that the equation (3.12) holds. □

4. Principal component analysis of dyadic stationary processes

Let $\{Y(t), t \in T\}$ be a q-dim. dyadic stationary process having a zero mean vector and a Walsh spectral density matrix:

$$f_Y(\lambda) = \Phi(\lambda) G \Phi(\lambda)', \quad 0 \le \lambda \le 1, \tag{4.1}$$

where $\Phi(\lambda)$ is a $q\times q$ matrix valued function of the form

$$\Phi(\lambda)=\sum_{l=0}^{p} A_l W(l,\lambda),\quad 0\leqslant\lambda\leqslant 1,$$

with a non-negative integer $p=2^m-1$ and G is a $q\times q$ positive semi-definite matrix. We note that under such conditions as shown in Theorem 3.1 every dyadic stationary process in a class of finite parametric linear models has a Walsh spectral density matrix of the form (4.1).

Now, let us consider the following problems:

(a) To find the s-dim. principal component process $X(t)=(X_1(t),X_2(t),\dots,X_s(t))'$, $s\leqslant q$, $t\in T$, of the process $\{Y(t),t\in T\}$, that is, a process $\{X(t),t\in T\}$ being such that (i) each component process $X_j(t)$ is a linear combination of $\{Y(t),t\in T\}$, (ii) the component processes $\{X_j(t),t\in T\}$, $j=1,2,\dots,s$, are mutually un-crosscorrelated and (iii) for each frequency λ, the j-th principal component process $X_j(t)$ has the j-th largest Walsh spectral density.

(b) Through the principal component process $\{X(t),t\in T\}$, to construct the best approximation $\tilde{Y}(t)$ to $Y(t)$ of the form

$$\tilde{Y}(t)=\sum_{n=0}^{\infty} H(n)X(t\oplus n),\quad t\in T,$$

where $q\times s$ coefficients matrices $\{H(n)\}_{n=0,1,2,\dots}$ are to be determined so as to minimize the "degree of nearness":

$$J(H,X)=E\{\|Y(t)-\tilde{Y}(t)\|^2\}.$$

Now, let $\mu_j(\lambda)$ be the jth largest latent root of $f_Y(\lambda)$ and $V_j(\lambda)$ the corresponding normalized latent vector of $f_Y(\lambda)$, and put

$$V(\lambda)=[V_1(\lambda)\quad V_2(\lambda)\quad\cdots\quad V_s(\lambda)].$$

From the rationality of the Walsh spectral density matrix, we note that

$$V(\lambda)=V(j/(p+1))\quad\text{for }\lambda\in[j/(p+1),(j+1)/(p+1)),\ j=0,1,2,\dots,p.$$

Hence, it is seen that the Walsh-Fourier coefficients of $V(\lambda)$,

$$H(k)=\int_0^1 V(\lambda)W(k,\lambda)\,d\lambda,\quad k=0,1,2,\dots,$$

vanish for all k greater than p; that is,

$$H(k)=0\quad\text{for all }k\geqslant p+1.$$

Thus, $V(\lambda)$ can be written as

$$V(\lambda)=\sum_{l=0}^{p} H(l)W(l,\lambda),\quad 0\leqslant\lambda\leqslant 1.$$

The solution of the principal component analysis stated above is given by the following theorem.

Theorem 4.1. *The principal component process* $\{X(t), t \in T\}$ *of* $\{Y(t), t \in T\}$ *satisfying the conditions* (i), (ii) *and* (iii) *in* (a) *is given by*

$$X(t) = \sum_{n=0}^{p} H(n)' Y(t \oplus n), \quad t \in T. \tag{4.2}$$

The Walsh spectral density matrix $f_X(\lambda)$ *of the principal component process* $\{X(t), t \in T\}$ *is diagonal and of the form*:

$$f_X(\lambda) = \mathrm{diag}[\mu_1(\lambda), \quad \mu_2(\lambda) \cdots \mu_s(\lambda)]. \tag{4.3}$$

The best approximation of $Y(t)$ *through the principal component process* $\{X(t), t \in T\}$ *is given by*

$$\tilde{Y}(t) = \sum_{l=0}^{p} H(l) X(t \oplus l), \quad t \in T, \tag{4.4}$$

and the minimum value achieved is

$$J_{\min} = \sum_{j=s+1}^{q} \int_0^1 \mu_j(\lambda)\, \mathrm{d}\lambda. \tag{4.5}$$

Proof. Let the Walsh spectral representation of $\{Y(t), t \in T\}$ be

$$Y(t) = \int_0^1 W(t, \lambda)\, \mathrm{d}Z_Y(\lambda).$$

Each increment $\mathrm{d}Z_Y(\lambda)$ of the Walsh spectral process has a zero mean vector and the covariance matrix $f_Y(\lambda)\, \mathrm{d}\lambda$. Applying the results of Darroch (1965) to $\mathrm{d}Z_Y(\lambda)$, we can see that the principal component of $\mathrm{d}Z_Y(\lambda)$ is given by

$$\mathrm{d}Z_X(\lambda) = V(\lambda)'\, \mathrm{d}Z_Y(\lambda) = \sum_{l=0}^{p} H(l)' W(l, \lambda)\, \mathrm{d}Z_Y(\lambda),$$

and also the best approximation to $\mathrm{d}Z_Y(\lambda)$ through $\mathrm{d}Z_X(\lambda)$ by

$$\mathrm{d}\tilde{Z}_Y(\lambda) = V(\lambda)\, \mathrm{d}Z_X(\lambda) = \sum_{l=0}^{p} H(l) W(l, \lambda)\, \mathrm{d}Z_X(\lambda)$$

in a sense that it minimizes the "degree of nearness" at λ:

$$J_\lambda = E\{\|\mathrm{d}Z_Y(\lambda) - \mathrm{d}\tilde{Z}_Y(\lambda)\|^2\}.$$

Here we can see that the minimum value achieved is given by

$$\min J_\lambda = \sum_{j=s+1}^{q} \mu_j(\lambda)\, \mathrm{d}\lambda. \tag{4.6}$$

The principal component process $\{X(t), t \in T\}$ in the time domain is given by

$$X(t) = \int_0^1 W(t, \lambda)\, \mathrm{d}Z_X(\lambda) = \sum_{l=0}^{p} H(l)' Y(t \oplus l).$$

The Walsh spectral density of $\{X(t), t \in T\}$ is easily shown to be

$$f_X(\lambda)\,\mathrm{d}\lambda = E\{\mathrm{d}Z_X(\lambda)\,\mathrm{d}Z_X(\lambda)'\} = \operatorname{diag}[\mu_1(\lambda) \quad \mu_2(\lambda) \quad \cdots \quad \mu_s(\lambda)]\,\mathrm{d}\lambda.$$

Also, we see that the best approximation of $Y(t)$ is given by

$$\tilde{Y}(t) = \int_0^1 W(t,\lambda)\,\mathrm{d}\tilde{Z}_Y(\lambda) = \sum_{l=0}^{p} H(l)X(t \oplus l), \quad t \in T,$$

since the degree of nearness J is an integral of J_λ with respect to λ over [0, 1] and can be written as

$$J = E\{\| Y(t) - \tilde{Y}(t)\|^2\} = \int_0^1 E\{\mathrm{d}Z_Y(\lambda) - \mathrm{d}\tilde{Z}_Y(\lambda)\|^2\}.$$

Thus, by summing up the value (4.6) given above for all λ, $0 \leq \lambda \leq 1$, we see that the minimum value achieved is obtained as (4.5). □

5. Concluding remarks

Though not reported here, the canonical correlation analysis between two sets of jointly dyadic stationary processes with rational Walsh spectral density matrices can be easily carried out with the same approach as in the principal component analysis. Also in the canonical correlation analysis in this situation, similar results to those in the principal component analysis are obtained, such as each canonical component process being obtained as a one-side finite sum of the corresponding process. Details for the canonical correlation analysis of dyadic stationary processes can be obtained from the authors on request. We note that the principal component process as well as the canonical process based on the approximating Walsh spectral ensity matrix of rational type are constructed as one-side finite sums of the observed processes and that the number of terms used in the principal component process as well as in the canonical component process corresponds to the degree of the approximation in the Walsh spectral density. These properties are also retained for p-adic stationary processes; that is, random processes whose covariance matrices are invariant under the shift defined by p-adic addition.

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